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Introducing Equations in Early Algebra

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Abstract

The overwhelming presence of a procedural meaning of equality and equations reported in previous research has led to a call for suitable pedagogical interventions to nurture a relational meaning of these concepts. This paper is a response to that call. Drawing on the theory of objectification, the first part deals with the configuration of a Grade 3 (8–9-year-old students) teaching-learning activity that seeks to create the classroom conditions for the formation of the mathematical operations and operation-based rules that underpin the algebraic simplification of linear equations. Instead of using problems involving abstract open arithmetic sentences or alphanumeric equations (e.g., $5 + \_ = 16$; $2n + 3 = 11$), the teaching-learning activity resorts to story-problems. Two visual semiotic systems serving to model and solve the story-problems were devised. The story-problems were framed in narratives that allowed the teacher and the students to infuse equations, their equating parts, and the mathematical operations with contextual meanings. The first part of the paper includes the theoretical assumptions about the teaching-learning activity and its configuration, and a rationale behind the devisal of the semiotic systems. The second part presents a Vygotskian multimodal genetic analysis of the teaching-learning activity; that is, an analysis that shows the formation of concepts *in motion*, in the process of their genesis. The genetic analysis sheds some light on the way students, in their work with the teacher, encountered and refined the cultural-historical algebraic meanings of the equal sign and equations, and the concepts required in solving equations.

Keywords Algebra • multimodal semiotics • Vygotsky • gestures • algebraic operations • story-problems

In memory of Eugenio Filloy

1. Introduction

Equality is one of the most fundamental relations in mathematics. A = B asserts something about A and B: that they are 'equal' or 'equivalent' in a certain sense—not necessarily in the sense of being identical. A is equal to A, but there may be other objects different from A that may make the relation A = B true. For instance, in $\mathbb{Z}_3$, 4 = 4, but 4 = 13 too (as 4 is congruent to 13 mod 3). More generally, A and B can be seen as signs or representations of a *same* object (a numerical object or a geometric object, for example) in the sense of a given '=' equivalent relation. Asghari (2019) has described the lengthy and painful process that mathematicians underwent in the early 20th century to clarify the equality/equivalence concept, which "is beset with a lot of riddles and gives rise to challenging questions" (Otte & de Barros, 2013, p. 171).

The concept of numerical equality and the ensuing concept of equation are central parts of school mathematics. Several studies conducted with primary and middle-school students have focused on the identification of the meanings that these students ascribe to the equal sign and their understanding of equations. One of the main findings has been the identification of *procedural* and *relational* meanings of the equal sign. A procedural meaning leads to the conceiving of the equal sign as an inscription that prompts one to carry out a calculation. By contrast, a relational understanding leads to seeing the equal sign as referring to an attribute of *sameness* of the equated parts A and B in A = B (Kieran, 1981; McNeil et al., 2006).

It has been found that there is "a strong relation between equal sign understanding and success in solving equations" (Knuth et al., 2006, p. 308). For instance, Stephens et al. (2013) found that "students in the elementary and middle grades tend to view the equal sign operationally and that very few demonstrate a strong structural sense of equations" (p. 181). Thus, in an often-quoted example, to solve the equation $8 + 4 = \square + 5$, Carpenter et al. (2003, p. 9) found that primary students frequently responded 12 or 17; these students hold a procedural understanding of the equal sign. And when one of the students was asked to give reasons, after being reminded that there is, however, an equal sign between $8 + 4$ and the rest, he said: "Yeah, but you have to add all the numbers. That's what it [the number sentence] says to do" (p. 11). In this case, the equation is comprehended in computational terms without a proper structural understanding of it. Students holding a relational meaning of the equal sign, by contrast, see the equated parts in a structural relation: an equated part is seen as being the same as the other equated part of the equation. These students may compute 8 + 4 and think of the right side of the equation as being equal to 12; or they can proceed to a "compensatory strategy" (Rittle-Johnson et al., 2011, p. 87), noticing for example that 5 is one more than 4; hence the sought-after number must be 1 less than 8.

The overwhelming presence of a procedural meaning of the equal sign and computational understandings of equations reported in previous research led Matthews and Rittle-Johnson (2009), Stephens et al. (2013), Jones et al. (2012), and other researchers to call for suitable pedagogical interventions in the classroom. There is a need

> for elementary and middle school teachers to focus on the meaning of the equal sign and equation structure on a regular basis and not assume that because the equal sign appears throughout students' mathematics experiences that the symbol is well understood . . . We know that simply telling students what the equal sign means does not effectively develop understanding (Stephens et al., 2013, p. 181).

The recent studies by Bajwa and Perry (2021) and Stephens et al. (2022) attempted to answer this call. Bajwa and Perry (2021) envisioned a tutor-student intervention with Grade 2 and 3 students based on a computer game pan-balance scale to favor a relational understanding of the equal sign. Stephens et al. (2022) resorted to physical balances in instructional contexts with Kindergarten to Grade 2 students. In both studies, the researchers assessed students' thinking through pre-, mid-, and post-intervention through questionnaires or interviews. The results in both studies are encouraging in that they show that instructional interventions may help young students understand equations and the equal sign in ways that are required in algebra. Yet, *genetic analyses* of concept of formation (i.e., analyses showing the very process of the genesis of the students' algebraic concepts as they emerge *in/during* instruction) is still needed. By investigating artifact- and symbol-use *in/during* the interaction between teacher and students and in the interaction among students, genetic analyses of classroom teaching-learning activity can show the complex emergence of mathematical concepts and contribute to better appraisal of the ways in which students' mathematical thinking is formed and transformed.

By offering a genetic analysis of classroom activity, in this paper the intention is to contribute to research on the teaching and learning of equations in early algebra. In the paper I adopt the

Vygotskian semiotic perspective of concept formation featured in the theory of objectification, a perspective that focuses on tracking the "progressive emergence of conscious awareness of concepts and thought operations" (Vygotsky, 1987, p. 185) in the interaction of teachers and students. Drawing on data from a longitudinal study, the goal is to shed some light on the way students in their work with the teacher encountered and refined the cultural-historical algebraic meanings of the equal sign and equations, and the concepts required in solving equations.

The paper consists of two main parts. The first part contains the theoretical framework and the adopted methodology. It includes a rational behind the devisal of two concrete semiotic systems that were pivotal in pedagogically introducing young students to a relational meaning of the equal sign and the algebraic simplification of equations. The second part presents some Grade 3 episodes that illustrate concept formation during the classroom processes of solving equations.

2. The theoretical framework

2.1 Equating things

The equal sign that we use today is a relatively recent invention. It was introduced by Robert Recorde in 1557 (Cajori, 1993). Before the equal sign was introduced, mathematicians used other signs (like a dash "—") or words with different meanings to talk about equating things. For example, medieval Arabic mathematicians used the following: *sawiya* to mean "to be equivalent, be equal"; *mithl* (very close to sawiya) to mean "similar, of the same kind"; *'adala* to mean "to be equal"; and the prefix *ka-* to compare things as in "like" or "as" (Oaks, 2010, p. 266). Of all these terms, medieval Arabic mathematicians used a specific one to convey the *relational* meaning of equal in algebraic equations, namely *'adala*. "*'Adala* was employed differently than the other words. It was used to equate the two sides of an algebraic equation" (p. 266). Oaks suggested that this use of *'adala* stems "from the fact that the word originated from an adjective meaning 'well balanced.' Scales are naturally suited for comparing two multitudes of objects" (p. 265). The origin of *'adala* comes in fact from metrology and a noun meaning "justice" (p. 271).

To better understand the difficulties that have been reported with primary and middle-school students concerning the meaning of the equal sign, it is worth noticing that *mithl* and its derivatives (e.g., *musāwin*), along with *sawiya,* were often used to assert equality or equivalence in the comparison of numeric or geometric objects. To express a numerical fact, such as 10 x 7 is 'equal to' 7 x 7 + 3 x 7, medieval Arabic mathematicians used *musāwin*.[1]

We see that, through different words, these mathematicians made clear a conceptual distinction between various aspects of equality. What today we term a procedural meaning of equality would fall into the *musāwin*/*sawiya* category. By contrast, the relational meaning of equality would fall into the *'adala* one. In many languages, the discernment of these different meanings is blurred and all of them collapse into one single word, namely, the word *equal*. This is the case in English.

2.2 The genetic analysis of concept formation

The central idea of a Vygotskian genetic analysis of concept formation is that such analysis needs to be carried out in strict connection with the *activity* that puts the concepts in motion. In the theory

[1] The 10th century mathematician Ikhwan al-Ṣafā' wrote, "So I say that the product of the ten by seven is equal to (*musāwin*) the product of seven by itself and three by seven" (Oaks, 2010, p. 267).

of objectification (Radford, 2021), in the context of school learning, this activity is the classroom teaching-learning activity.

The concept of teaching-learning activity has in the theory of objectification a very specific meaning. It does not amount to instructional task + classroom discourse, nor is it a mere set of coordinated actions of various individuals. Activity is an *evolving system* (an unfolding form of *energy*) dialectically embedding the flux of emotional, affective, social, and ethical relations, and intellectual, discursive, and material processes that individuals produce in dealing with a common problem or situation. In the case described in this paper, activity is the system that teacher and students produce together in dealing with the simplification of equations. In this activity—where teaching is learning, and learning is teaching—teaching and learning are not two separate activities but a same teaching-learning activity: the collective activity that puts into motion those concepts pertaining to the cultural way in which one solves equations.

It is against this theoretical understanding of activity that our research team devised a sequence of teaching-learning activities to foster young students' understanding of the concepts required to solve equations, in particular those required to *simplify* equations algebraically (e.g., concepts related to the equal sign, the algebraic concept of unknown, and the various operations involved in the deductive transformation of equation). Let us examine in more detail the concepts at stake.

The simplification of equations involves the production of a chain of equations, $E_1, E_2, \ldots, E_n$, where each equation is *deduced* from the previous one, leading to E_n: $x = \alpha$. The *deduction* from E_i to E_{i+1} is ensured by applying algebraic *rules* involving *operations* on determinate and indeterminate quantities. For instance, from $3x - 1 = x + 3$ we deduce $3x = x + 4$ by applying the rule of *adding* 1 to both sides of the equation; from the last equation we deduce $2x = 4$ by applying the rule of subtracting x from both sides of the equation; finally, from the last equation we deduce $x = 2$ by applying the rule of *dividing* both sides of the equation by 2. In the case of linear equations, the required operations are additions/subtractions and divisions/multiplications. From the genetic analysis of concept formation that we envision here, the students' understanding of these operation-based rules is what is at stake. The pedagogical problem is, then, to devise teaching-learning activities through which young students can encounter in meaningful ways the culturally and historically aforementioned algebraic rules underpinning the simplification of equations. In the theory of objectification, the configuration of teaching-learning activities rests on two pedagogical principles.

2.3 The pedagogical principles

The first organizing principle of our teaching-learning activities has to do with the creation of a social space of interaction and communication where students find room to interact among themselves and with the teacher. This interaction, however, is not seen in terms of 'negotiation of meanings', but as a cooperation among individuals in the classroom's collective production of ideas—what we term the production of a *common work*. In this line of thinking, teachers are not seen as helpers or coaches: like the students, teachers are engaged in the classroom production of ideas within the parameters of a division of labor.

The second organizing principle has to do with the collective production and circulation of ideas in the classroom. It deals with the identification and organization of the mathematical problems to be solved and how to solve them. Both principles are dialectically intertwined, yet they are not identical.

The first principle was operationalized through the formation of small groups to discuss how to solve equations and the creation of spaces for general discussions. The second principle was operationalized through the creation of two concrete semiotic systems. These semiotics systems were intended (a) to bring to the fore the elementary algebraic concepts required in solving equations, and (b) to prepare the students for their encounter with the alphanumeric symbolism (which happened in Grade 4 one year later).

2.4 The semiotic systems

As Blanton et al. (2018) noted, in the investigation of young students' understanding of the equal sign and equations, tasks are usually based on "written numeric symbols" (p. 171), that is, symbols for numbers and operators (e.g., 3, +). These tasks include open abstract number sentences—e.g., $8 + \square = 8 + 6 + 4$ (Rittle-Johnson et al., 2011, p. 91)—or abstract alphanumeric equations—e.g., $4m + 10 = 70$ (Knuth et al., 2006, p. 301). Because of their abstract, decontextualized nature and the signs they use, these number sentences are likely to induce computational understandings that might not be aligned with the algebraic understanding of the unknown, the equal sign, and the equation, that we expect in algebra. We were hence motivated to design *visual* concrete semiotic systems to represent the two kinds of mathematical 'species' required to model linear equations, namely, quantities and unknowns.

Our first system is the *Concrete Semiotic System* (CSS). It is comprised of the following material objects: a) paper envelopes that contain the same unknown number of cardboard cards, b) cardboard cards, and c) the equal sign. The design of this semiotic system was inspired by an idea about the unknown in algebra attributed to the most talented mathematician of the 13th and 14th centuries, Antonio de Mazzinghi. Mazzinghi belongs to the abacist Italian tradition that preceded symbolic algebra. The unknown was not represented by a letter; it was referred to as "*cosa,*" literally "a thing." Mazzinghi defines "a thing" as follows: "a thing is an occult quantity" (quoted by Franci & Rigatelli, 1988, p. 15). The envelope of the CSS is the metaphorical embodiment of Mazzinghi's idea.

Our second semiotic system is the *Iconic Semiotic System* (ISS). The ISS replaces concrete objects with iconic drawings {  , , =, ↑ }. The additional 'arrow' sign replaces actions performed on concrete cards or envelopes of the CSS during the process of simplifying equations. The arrow could be substituted by simple lines indicating that a card or envelope (or sets of these) are removed.

Instead of presenting the students with abstract equations, we used *story-problems* that were then translated and solved in the CSS and later in the ISS.[2] The range of story-problems that can be formulated in natural language and translated in the CSS is very limited, but it is enough to ensure that young students have their first encounter with equations and their algebraic simplification.[3]

[2] A story-problem is a sub-category of the category of word-problems. The difference is that a story-problem includes an experiential context and a narrative in which agents participate in some way or another. As described in the following sections, our story-problems include children (agents) having and/or receiving things (narrative) such as candies or cards (experiential context). A problem like "Divide 10 into two parts such that the division of one by the other gives 4" is a word-problem, not a story-problem.

[3] Our CSS is not far from balance models used sometimes in research and instruction (see, e.g., Blanton et al., 2018; Stephens et al., 2022; Vlassis, 2002; for a review, see Otten et al., 2020). In our case, the balance is replaced by story-problems that, through narratives, bring to life the equality of the equating parts.

3. Methodology

The data presented below come from a longitudinal research program carried out in a French public school in Sudbury, Canada. A class of 25 students was followed from Grade 2 to Grade 4. In Grade 2 (7–8-year-old students) the emphasis was placed on becoming familiar with translating simple story-problems into the CSS and the ISS and solving them in those semiotic systems.[4] Although equations of the type $ax + b = cx + d$ were used, the focus was put on $ax + b = c$ and $c = ax + b$ equations. In Grade 3, the students kept using the CSS and the ISS; the emphasis was put on the transition from equations of the type $ax + b = c$ to equations of the type $ax + b = cx + d$. The central goal was the understanding of the operation-based rules involved in the simplification of equations. In Grade 4, the class started using the standard algebraic alphanumeric semiotic system; at the end of the year, the students dealt with equations stated in the standard alphanumeric semiotic system and solved equations such as $3n + 3 = 1n + 9$.

To collect data, during the longitudinal program, the students were divided into groups of two or three. The activities were recorded using five video-cameras, each closely following the work of a student group. We also collected students' sheets and wrote field notes at the end of each activity.

In what follows, we focus on Grade 3. We report here the first of three consecutive teaching-learning activities devoted to algebra. Like all mathematics lessons in the school, the algebra teaching-learning activities lasted 100 minutes each.

In line with the idea of genetic analysis mentioned above, our unit of analysis was the classroom teaching-learning activity. After transcribing the videos, we performed a multimodal semiotic analysis on "salient episodes" (Radford, 2015) in accordance with the non-positivist interpretative methodological paradigm outlined by Radford (2021), and Radford and Sabena (2015). The multimodal semiotic analysis revolves around the role of signs, language, artifacts, and the body (e.g., gestures, actions, perceptual activity) in concept formation (Radford, 2014); it allows us to investigate the teachers' and students' joint processes of meaning-making during the classroom encounter with cultural-historical constituted algebraic knowledge.

During the research program, teachers worked closely with the research team. They attended research meetings where they had the opportunity to see videos of what students had accomplished in the previous years. They actively participated in task design and writing field notes. They taught the lessons.

The small groups reported below were selected as they were the best paradigmatic matches for the observed classroom phenomena.

4. Grade 3: The ideas behind the simplification of equations

The activity was divided into three parts, namely, (a) opening general discussion, (b) work in groups, and (c) closing general discussion.

4.1 The opening general discussion: The equation $3 + x = 7$

The teacher discussed with the whole class a story-problem in which a child, Sara, had an envelope containing some hockey cards. While the teacher showed the envelope to the class, she said: "The envelope is sealed. We don't know how many hockey cards are inside." After having stuck the

[4] The first contact with equations in Grade 2 was described by Radford (2017).

envelope to the board, she said: "But Sara already had 3 hockey cards," and stuck 3 cards to the board. Then the teacher told the class that Sara's friend, Christina, had 7 hockey cards, and stuck 7 cards beside Sara's envelope and cards. She drew an equal sign between the two groups of objects. She asked: "What is this symbol? What does it mean?" A student answered: "It means that Sara and Christina must have the same amount of hockey cards." Then the teacher said: "I would like that one student use what is on the board to try to figure out how many cards are in the envelope." Jase volunteered (all names are pseudonyms).

Figure 1.1 shows Jase and the equation in the CSS. Jase quickly found that there were 4 cards in the envelope, for, as he said, "4 (pointing to the envelope) plus 3 equals 7 (pointing to the 7 cards)."[5] The teacher praised Jase's solution and asked the class for a different way to solve the problem. Gustav circled a block of 3 cards on the left side of the equation and circled 3 cards on the right side. He drew a second equal sign to mean that the 3 cards on the left were equal to the 3 identified cards on the right. "So, this here (pointing to the remaining 4 cards) must be equal to this (pointing to the envelope; see Figure 1.2). The envelope must have 4 cards inside." The teacher also praised Gustav's idea and suggested that one could try to *isolate* the envelope. The idea of isolating the unknown was introduced in Grade 2 (see Radford, 2017), but as we can see, it is far from being the students' first choice in approaching the problem.

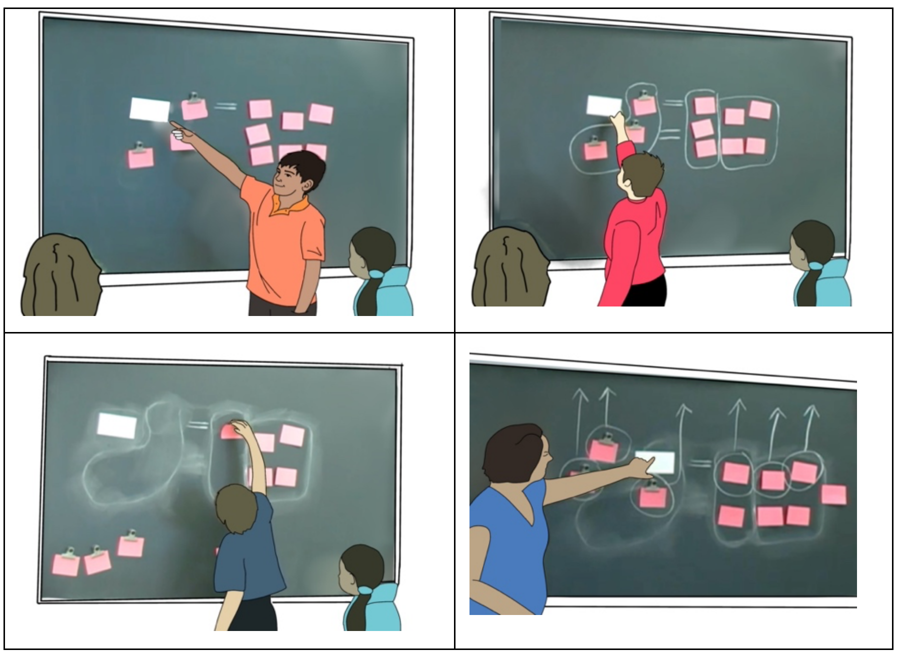

Fig. 1. Various problem-solving procedures to solve $3 + x = 7$

1. Teacher: What do we mean by isolate? If I tell you, I'd like us to isolate the envelope (*points to the envelope; several students raise their hands*). I'd like to isolate the envelope. Cyr?
2. Cyr: Does that mean like putting it alone?

[5] Figures are numbered from from left to right and top to bottom. In Figure 1 they are numbered as 1.1, 1.2, 1.3, 1.4.

3. Teacher: Putting the envelope alone. That's exactly what it means. How could we put the envelope alone? We want to know how many cards are in the envelope, but we want to isolate the envelope. How can we put the envelope alone, isolate it? What do you think?
4. Cyr: (*Cyr comes to the board and starts removing cards*).
5. Teacher: Okay, wait . . . let's remove them one at a time, okay? We remove one, but that (*pointing to the equal sign*) says "equal"; that the amount of cards on both sides is equal; then, if you remove one [card] on this side, what do you do?
6. Cyr: I remove another one from there (*he removes a card from Christina's side*).
7. Teacher: You have to remove one from this (*pointing to the right*) side.
8. Cyr: Then you'd remove another one from here (*the left side*) . . . Another one over here (*the right side*). The last card here (*left side*) and one last card from here (*right side; see Figure 1.3*) and then there's . . .
9. Teacher: And then what happens? Is your envelope isolated? Is the envelope alone?
10. Students: Yes!
11. Cyr: Then, I would count how many cards there are … 4!
12. Teacher: That means that the envelope (*pointing to the envelope*) equals (*points to the equal sign*) . . . how many hockey cards?
13. Mariana: 4!

This passage from the opening general classroom discussion shows how the teacher and the students contributed to the collective production of ideas. The students brought to the fore different meanings of the equal sign. Jase capitalized on his familiarity with adding small numbers. Reading the equal sign in a procedural sense (which evokes the medieval Arabic sense of *musāwin*/*sawiya*), the answer came without difficulty. Let us call Jase's procedure the Computational Procedure.

Gustav proceeded in a different way. He perceptually recognized equal chunks of signs on both sides of the equation. From this perceptual recognition he *compared* the equal to the equal and *associated* the remaining parts of the diagram. This procedure, which is based on a *relational* understanding of the equal sign (evoking the medieval Arabic sense of *'adala*), led him to *deduce* the value of the envelope. He said: "So, this here (pointing to the remaining 4 cards) *must* be equal to this (pointing to the envelope)." Let us call Gustav's procedure the Comparison Procedure. The procedure is certainly powerful. Its perceptual heuristic nature avoids the process of equation simplification. Indeed, it avoids operating on/with known and unknown quantities, which makes it very different, cognitively speaking, from the isolating-the-unknown procedure. The latter came to the fore out of the contribution of Cyr and the teacher. Let us have a look at how Cyr and the teacher did it.

In Line 1 the teacher talks about "isolating" the envelope. She *names* the problem-solving strategy. Then, she asks for its *meaning*. In Line 2 Cyr's answer comes in an interrogative form: "Does that mean like putting [the envelope] alone?" In Line 3 the teacher confirms Cyr's idea. Isolating means putting the envelope alone. Then, the teacher turns the focus on *how* to do that, reminding the class of the overarching goal of the isolating procedure: "We want to know how many cards are in the envelope . . . "—but not by guessing or counting. Therefore, she immediately adds: ". . . but we want to isolate the envelope." The *how* question is intended to bring the students' attention to the mathematical *operations* to be performed in the simplification of the equation.

In Line 4 Cyr goes to the board and, without talking, starts removing cards. He *shows* the procedure through *embodied actions*. In Line 5 the teacher asks him to wait and says: "Let's remove them one at a time." She mentions the *name* of the operation that helps simplify the equation. The name is *removing*. Naming something makes the named thing an explicit object of discourse; it makes communication easier. But it also makes the named thing salient. Indeed, through a word, the named thing becomes a more precise object of consciousness and thought: "speech is not only an instrument for communication but also an instrument of thought; consciousness develops chiefly with the help of speech" (Vygotsky, 1993, p. 89).

However, the named operation (removing) still needs to be recognized as part of a simplifying *rule*. This is what the teacher accomplishes in Line 5. Indeed, in the last part of her utterance (Line 5), the teacher brings the students' attention to the *relational* meaning of the equal sign ("the amount of cards on both sides is equal"). For the new equation to continue to be valid, the same operation must be performed on the other side of the equation. The teacher does not say this; she asks the question to involve the class in the collective production of ideas. In Line 6 Cyr answers the question. He applies the rule in Line 8. From Line 4 to Line 8 there is a shift from action alone to action-and-language that is crucial in the process of concept formation.

Let us notice that the established rule, $\mathcal{R}$, is about removing *one* card at a time. To distinguish it from other rules, we shall write $\mathcal{R}$ and, in brackets, the actions/operations associated with the rule. In this case, we shall write $\mathcal{R}(r[1c])$, where *r* stands for the operation, which is removing (so r = removing), and $1c$ to indicate that the number of cards removed from the equated sides of the equation is 1 card.

Schematically speaking, the equation was solved by applying the rule $\mathcal{R}(r[1c])$ three times:

$$E_1 \xrightarrow{\mathcal{R}(r[1c])} E_2$$

$$E_2 \xrightarrow{\mathcal{R}(r[1c])} E_3$$

$$E_3 \xrightarrow{\mathcal{R}(r[1c])} E_4$$

where E_4 is $x = 4$.

After Cyr's intervention, the teacher summarized the main ideas behind isolating-the-unknown procedure. In preparation for the upcoming work on the ISS, the teacher took advantage of the general discussion to introduce a new sign: a contour-and-arrow sign that can be used to signify, when drawing equations, that objects have been removed. The teacher explained: "[removing this card] is the same as if you take this card here (*she circles a card on the left)* and you take it away (*she makes an arrow over it*), right?" (See Figure 1.4). The mathematical rule that in the CSS appeared as the *bodily action* of eliminating things on both sides of the equation now has a written sign in the ISS.

4.2 Work in groups: The equations $x + 2 = 8$ and $6 = 4 + x$

After the general discussion the students worked in small groups. The first two problems featured the equations $x + 2 = 8$ and $6 = 4 + x$ in the CSS (see Figures 2.1 and 2.2). Although the responses were quickly obtained by the Comparison Procedure in the CSS, as Gustav did before (Figure 1.2), the students used the Isolation Procedure when they drew the equation and solved it in the ISS (see Figures 2.3 and 2.4).

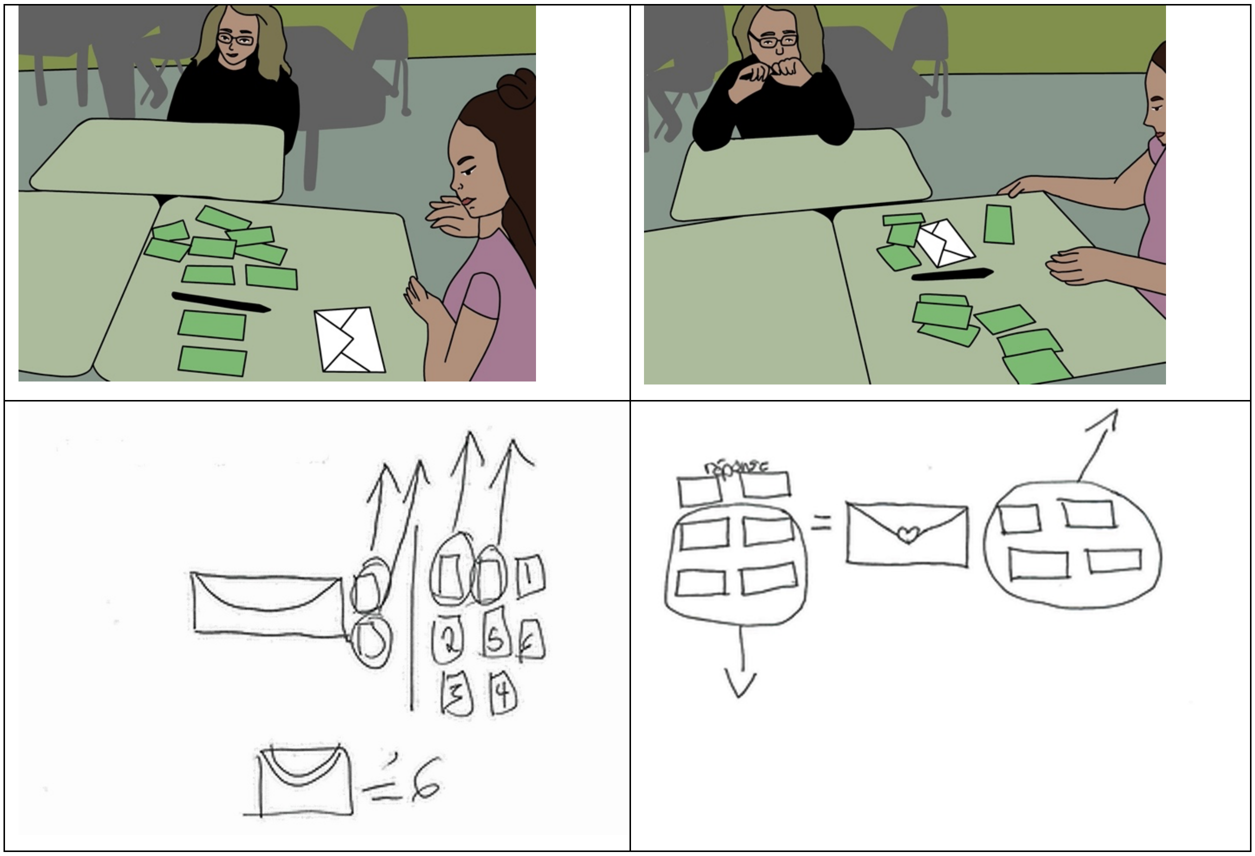

Fig. 2. Solving $x + 2 = 8$ and $6 = 4 + x$ in the CSS and the ISS

The teacher came to see the group. Referring to the $6 = 4 + x$ equation, Elsa explained the procedure: "You have to circle all the hockey cards (*she points to the 4 cards on the right side*), and you have to remove them, and the cards on the other side, so in the envelope you have 2 [cards]."

We see that the removing operation is named, and that the operation-based rule is applied.

In terms of our genetic analysis, it is worth noticing that, in the CSS, the students did not use the card with the equal sign that they were provided with along with the envelopes and hockey cards. It was already mentioned that, before the invention of the equal sign, mathematicians used other signs, like a dash. Here, the students used a pen to divide the equating parts (Figures 2.1 and 2.2). In Figure 2.3 we see two signs to represent the equality: a vertical line and the usual equal sign. In Figure 2.4 the students resort to only the usual equal sign. In all these equations, the pencil, the vertical line, and the written equal sign are synonymous: all bear the expected relational meaning of equality. But there is still more: there is also a refinement of the removing operation that allows the students to simplify the equation. In the opening general classroom discussion, the operation-based rule was applied to one object at a time—the $\mathcal{R}(r[1c])$ rule. This is what Cyr and the teacher did in Figures 1.3 and 1.4, and what the students did in Figure 2.3 too. In Figure 2.4, however, a new idea appears: we see that they removed four cards at once. There is a generalization of the rule: several cards can be removed *at once*. Let n be the number of cards removed simultaneously from both sides of the equation. The generalized rule can now be expressed as $\mathcal{R}(r[nc])$.

Schematically, we have the following:

$$E_1 \xrightarrow{\mathcal{R}(r[nc])} E_2.$$

4.3 Work in groups: The equation $2x + 1 = 6 + x$

Following a similar story-problem of two children having cards and envelopes, the Grade 3 students were presented with more complex equations in the ISS: the equations $2x + 1 = 6 + x$ (Figure 3.1) and $3x + 1 = 5 + x$ (Figure 3.2). I discuss the former in this section and the latter in the next section.

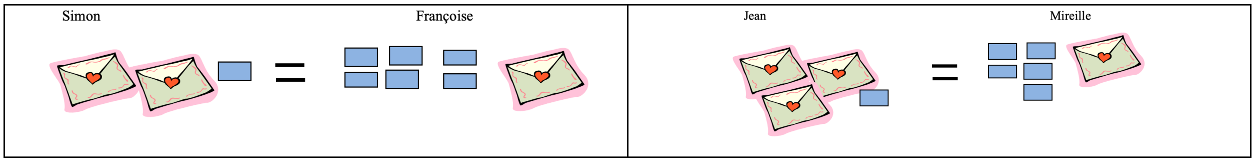

Fig. 3. The equations $ax + b = cx + d$ as presented to the students in the ISS

The students were asked to make and solve the equation in the CSS and in the ISS.

Before starting to solve the equation, Elsa commented: "Ça, ça va être super difficile!" ["This will be very difficult!"]. Indeed, Elsa's intuition is confirmed by the pioneer work of Filloy and Rojano (1989) who showed that equations of the form $ax + b = cx + d$ are more difficult than equations of the form $ax + b = c$, as now the Computational Procedure and other arithmetic procedures are more difficult to apply.

The students constructed the equation in the CSS (Figure 4.1). Although there was no equal sign, the students separated the equating parts carefully. The attention was primarily put on the *equating parts*, with the equal sign running in the background, so to speak (see Schwarzkopf et al., 2018). Then, the students drew the equation in the ISS. Here the equal sign appeared. Elsa said: "We must remove that (*she circles the card on the left side of the equation*) so that there are just envelopes, do you remember? (*then she removed one card on the other side*) 1, 1" (see Figure 4.2). Applying the $\mathcal{R}(r[1c])$ rule, they deduced a second equation. However, instead of continuing to simplify, they resorted to the Comparison Procedure.

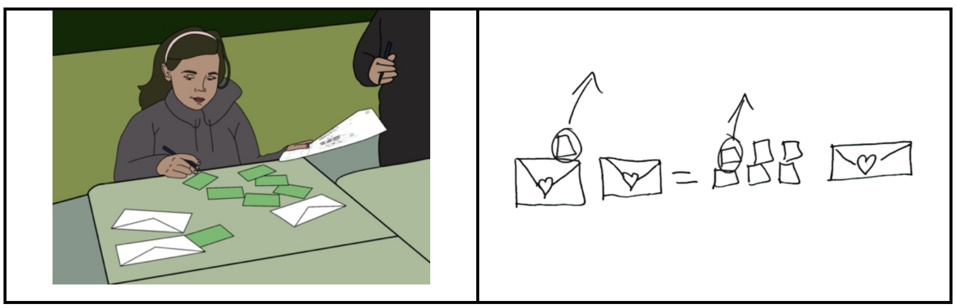

Fig. 4. Solving the equation $2x + 1 = x + 6$ in the CSS and the ISS

Schematically, we have the following:

$$E_1 \xrightarrow{\mathcal{R}(r[1c])} E_2$$

$$E_2 \xrightarrow{Comparison\ Procedure} response$$

The teacher arrived and asked the students to explain their procedure. The students constructed the equation in the CSS again. They removed one card from each side of the equation. The teacher said: "You are in the process of isolating! … How many envelopes do you want on one side?"

Puzzled by the question, the students looked at each other. A moment ago, Elsa had mentioned the idea of having envelopes on one side. The idea, however, was abandoned when they applied the Comparison Procedure. The teacher's intervention brought the idea back to life. On the one hand, the teacher acknowledged that the students were in the process of isolating the unknown. On the other hand, she raised a question that dealt with something that had not been thematized yet: the application of a *rule* to go from $2x = x + 5$ to x being equal to something. As in the previous cases, the rule is about removing. However, the argument of the rule is no longer cards, but *envelopes*. The algebraic resolution of equations of the form $ax + b = cx + d$ "involves operations drawn from outside the domain of arithmetic—that is, operations on the unknown" (Filloy & Rojano, 1989, p. 19).

1. Teacher: You want to know how many cards there are in ONE envelope (*she points to the envelope several times when she says ONE*) … First of all, you did this (*she removes a card from each side*) . . . you removed a card … Okay, what happens now? There are 2 envelopes (*pointing to the envelopes on one side of the equation*), then (*pointing to the objects on the other side of the equation*) 1 envelope and 5 cards.

2. Cora: We counted all these (*points to the cards*). It's 5. So, it (*pointing to one of the envelopes*) should have 5 too (*see Figure 5.1*).
3. Teacher: How do you know?
4. Elsa: We are going to remove (*she removes one envelope from the left side; see Figure 5.2*).
5. Teacher: You're removing 1 envelope?
6. Elsa and Cora: Yes. (*Elsa removes one envelope from the other side as well; Figure 5.3*).
7. Teacher: Why did you choose to do that?
8. Cora: Because these (*the sides of the equation*) must be equal.
9. Elsa: Because we must remove; because there must be only 1 envelope left (*she takes the envelope that is left*).
10. Teacher: Is it okay to remove 1 envelope and then 1 envelope? Is your equation still equal?
11. Cora: Yes!

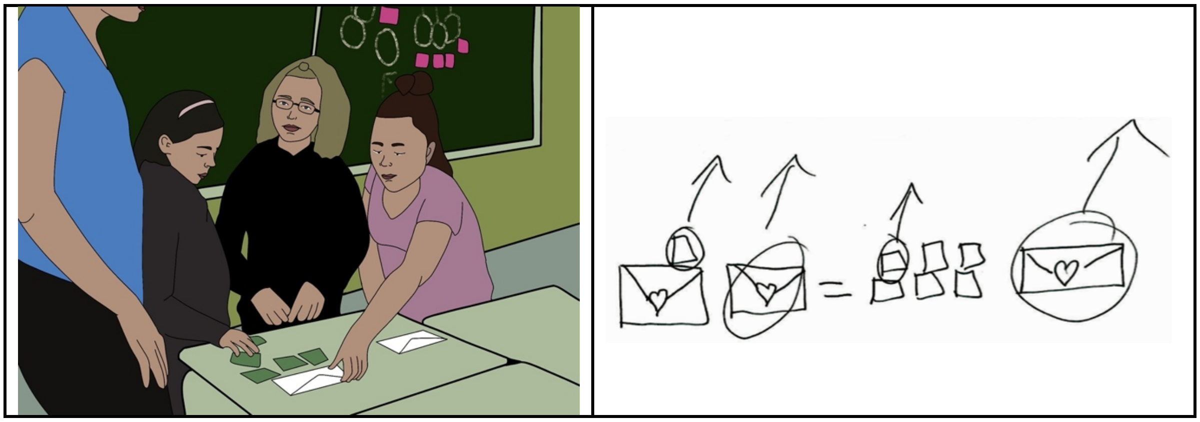

Fig. 5. Left to right, the teacher, Mia, Cora, and Elsa discussing the equation $2x + 1 = 6 + x$

In Line 1 the teacher started simplifying the equation as the students had done. In an encouraging tone, she asked: "What happens now?" In Line 2 Cora resorted to the Comparison Procedure, but the verbal articulation of ideas left important relations unaccounted for. These were the relations that the teacher asked for in Line 3. In Line 4 Elsa started removing one envelope from each side. The teacher wanted to make sure that the students understood the emerging rule. So, in Line 7, she asked for reasons. In Lines 8 and 9 the students offered two answers: Cora's focused on the conservation of the equality between both sides of the equation (the relational concept of the equal sign) while Elsa's focused on the idea of ending up with one envelope. In Line 10, the teacher wanted again to make sure that there was a clear understanding of the new emerging rule to simplify the equation. When the teacher left, the students came back to the equation in the ISS and removed one envelope from each side (Figure 5.4). Let us term the new rule $\mathcal{R}(r[1e])$ as it involves removing (r) one envelope (e) from each one of the equated parts.

The genetic analysis shows the lengthy process of conscious awareness of algebraic operations and rules, an awareness that, as we see, is being forged out of the joint work of the teacher and the students.

4.4 The equation $3x + 1 = 5 + x$

The students turned to the next problem, the one about the equation $3x + 1 = 5 + x$. They constructed the equation in the CSS and, instead of solving it with the help of concrete materials, went directly to the ISS to solve it. After drawing the equation, Cora started by removing one envelope from each side (the $\mathcal{R}(r[1e])$ rule). After that, she removed one card from each side (the $\mathcal{R}(r[1c])$ rule; see Figure 6.1).

12. Elsa: You only removed 1, but there must be only 1 envelope left. That's a problem (*they think for a while; then Elsa continues*). Four [cards], but there's not another envelope here (*points to the right side of the equation*).
13. Cora: There are 4 cards left, that's 4, we must remove these cards (*she circles the four remaining cards on the right side of the equation*) . . . And here (*she points to one of the remaining envelopes on the left side of the equation*) there are 0 [cards].
14. Elsa: Yes, but look! If there is 0 [cards] in the envelope, this (*pointing to the envelope on the right side of the equation*) will be 4 and this (*pointing to an envelope on the left side*) will be 1 [meaning perhaps zero]. But the 2 [envelopes] must be the same (*she points to the drawing*), the 2 [envelopes] must have the same number [of cards].

15. Cora: (*Explaining the idea again*) We removed that (*the four cards*).
16. Elsa: Then, there are 0, but there must be some cards [in the envelope].
17. Cora: Why?
18. Elsa: Here you have to remove this, here you remove this (*points with her pen to her drawing*) and you can't remove that [the 4 cards on the right side], because there are not 4 other [cards] here [on the left side] that you can remove . . .

The application of rules $\mathcal{R}(r[1e])$ and $\mathcal{R}(r[1c])$ led the students to the simplified equation $2x = 4$. Here the students found themselves in a new situation. While in the previous problem removing the same numbers of cards and envelopes was sufficient to isolate the unknown, in this problem the 'removing' operation is not enough. They could not continue removing envelopes for, as Elsa noted in Line 12, there were no more envelopes to remove on the right side. And "That's a problem." Cora suggested removing the four cards on the right side, which would lead them to zero cards. She then assigned zero cards to one of the two envelopes on the left side, which meant that there were four cards in the other envelope. Elsa pointed out two problems with Cora's suggestion. First, she argued that all envelopes must have the same number of cards (Line 14). Second, simplifying entails removing same things *on both sides* of the equation (Line 18). This requirement or condition was violated.

The students reached an impasse. "On est en train de se chicaner pour la réponse!" ["We are having an altercation over the response!"]. They tried to call the teacher, but she was busy discussing with another group. I was videotaping this group; I removed my headphones and went to talk to the students. I suggested that they use the concrete material (envelopes and cards). The students constructed the equation again and proceeded to remove one card and one envelope on each side.

19. Elsa: There are still 2 envelopes left (*see Figure 6.2*).
20. Mia: Then, there are 2 (*pointing to two cards*) here (*pointing to one of the envelopes*) and 2 (*pointing to the two remaining cards*) here (*pointing to the other envelope; see Figure 6.3*).
21. Cora: There must be 1 envelope!
22. Elsa: (*She removes one envelope and moves the cards to the other side of the equation; see Figure 6.4*)

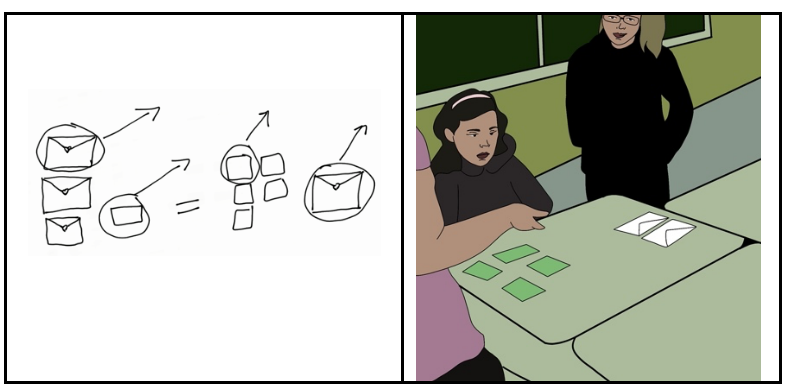

Fig. 6. Discussing the solution of $3x + 1 = 5 + x$ in the CSS

In Line 20 Mia suggested an idea. However, the idea was not taken into consideration by the other students, perhaps because the idea was not framed within the kind of actions and rules that the students recognized as legitimate in solving the equation. Yet, we see in Figure 6.4 that Elsa, in despair, removed one envelope and transferred the cards to the other side of the equation, placing them below the envelope, twice breaking the "do the same on both sides" rule. In despair, Elsa (like Cora in Line 13) stepped outside the boundaries of the algebraically thinkable that they had established so far. As Mikhailov once noted in a more general context, "rules themselves are not arbitrary and cannot be broken without detriment to the meaning of the idea that is to be conveyed" (Mikhailov, 1980, p. 213). The students' situation once again became very tense as we had seen in Line 18. Elsa said that they were still altercating and laughed. Laughing helped to dissolve a bit of the tension. Cora said: "Okay. We'll do it again!" They removed one card and one envelope from each side of the equation.

23. Elsa: There are 4 [cards]. We must have just 1 envelope remaining. So, we must remove 1 [envelope]; we don't have a choice (*she removes the envelope*).
24. Cora: Yes, but if we remove 1 . . . we must remove something else (*she points to the other side of the equation*).

They discussed for a while and came back to the simplified equation ($2x = 4$). After having looked attentively at the four cards and the two envelopes, Elsa said that she had an idea:

25. Elsa: Wait, wait! Here's my idea. Because we have 2 [cards] here . . . (*with each hand, she takes two cards from the bunch of four cards; then, she moves the two hands holding the cards and puts them in front of each of the envelopes; see Figure 7.1. When the cards arrive at their destination, she says*) . . . 2 in each envelope.

As if she could not believe her idea, she immediately started the body-language-artifact explanation again. She slid the four cards to where they were before, on one side of the equation. She said:

26. Elsa: Separate this [the 4 cards] into 2 . . . (*as she says this, she separates the cards; see Figure 7.2. Then she slides them to place them in front of each envelope*) . . . there are 2 in each envelope.

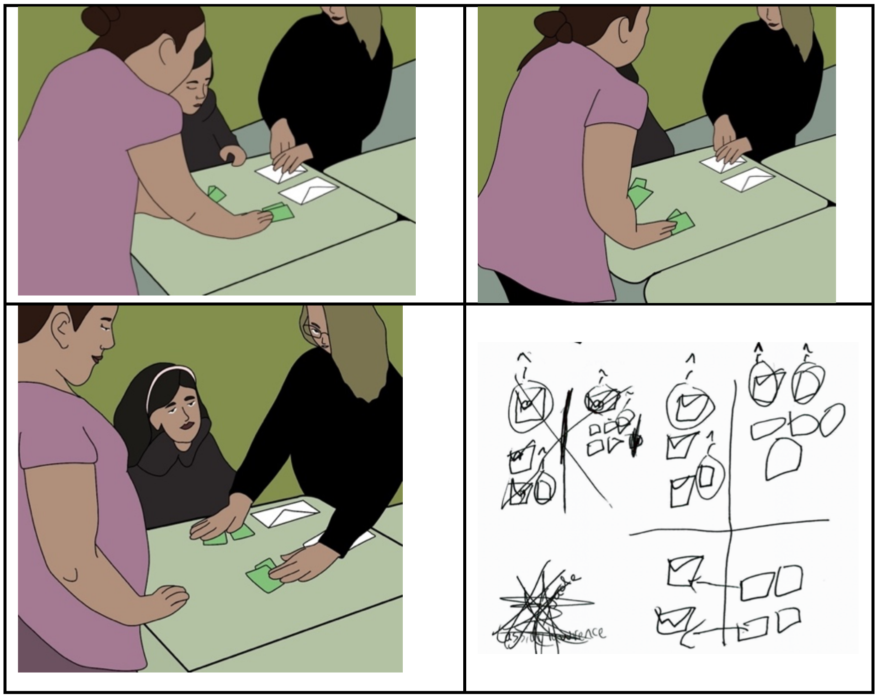

Fig. 7. Finding (again) how to solve the $2x = 4$ equation

Elsa's demonstration was followed by Mia's reaction:

27. Mia: This is what I said before, but you, you were . . .
28. Elsa: (*completing Mia's sentence*) . . . altercating!
29. Mia: . . . you said, no, no . . .
30. Elsa: I am sorry, Mia!

Cora made the equation again and went through the steps to isolate the unknown. When she reached the equation $2x = 4$, she said:

31. We are going to separate . . . (*and slides the two cards towards one envelope and two cards towards the other envelope; see Figure 7.3*).

Mia is right in arguing that she had suggested long before (Line 20, Figure 6.3) that each envelope had two cards. However, her suggestion was not articulated in terms of an *operation*, that of separating the cards. Rather, she followed a Comparison Procedure. Elsa did introduce an operation. In Line 25, the operation first appeared in an embodied way. The few uttered words were accompanied by a complex set of grabbing and sliding actions that remained unqualified linguistically. The linguistic articulation appeared when she again started the process of solving the problem. She said: "Separate this into 2, there are 2 in each envelope." Although the importance of the kinesthetic dimension that accompanied the emergent operation-based rule did not disappear, the thematic articulation in language was much more sophisticated. The new mathematical operation is *named*—'to separate'. This new operation is a precursor of what will later be the algebraic operation of division. Let us term the emergent rule $\mathcal{R}(s[c, e])$.

The students kept solving the equation in the CSS with their hands several times. It seemed that seeing was not enough and that feeling with their hands and their bodies was necessary. Then they drew their solution in the ISS. The new separating operation required a sign to be expressed in that semiotic system. Figure 7.4 shows that the students chose an arrow, which is reminiscent of the sliding action that made the two cards correspond to each envelope. In this sense, the sign is an icon of the action.

The students have now encountered the rules to simplify linear equations with (positive) known and unknown quantities. The first rules rest on removing equal species from both sides (known and unknown quantities). The medieval Arabic mathematicians subsumed these rules under the term *al-muqābala*. The rule that appeared in solving the simplified equation $2x = 4$ rests on decreasing the number of unknowns through an equal separation (division) of different species; these mathematicians termed it *al-radd* (Oaks & Alkhateeb, 2007) These rules respond to the question that the teacher asked during the general discussion mentioned in Line 3, section 4.1: *how* to isolate the envelope.

4.5 Work in groups: Inventing a story

In the next problem, the students were invited to invent a story similar to the ones seen previously, to translate it into an equation, and to solve it.

The story they produced reads as follows: "Martine has 10 (crossed out text) sticker (sic) she receives one envelope (crossed out text) for her birthday. Cas has 6 sticker (sic) she receives two envelope (sic) for Christmas. How many sticker (sic) do the two girls has (sic) if both (crossed out text) = the same" (see Figure 8.1, top).

After they wrote the story, they made a translation in the CSS (see Figure 8.2). They solved it quickly in the CSS by removing 6 cards from each side and 1 envelope from each side (using hence the $\mathcal{R}(r[nc])$ and the $\mathcal{R}(r[1e])$ rules). Then, they drew the solution in the ISS (Figure 8.2, bottom), where the equal sign is represented by a vertical line.

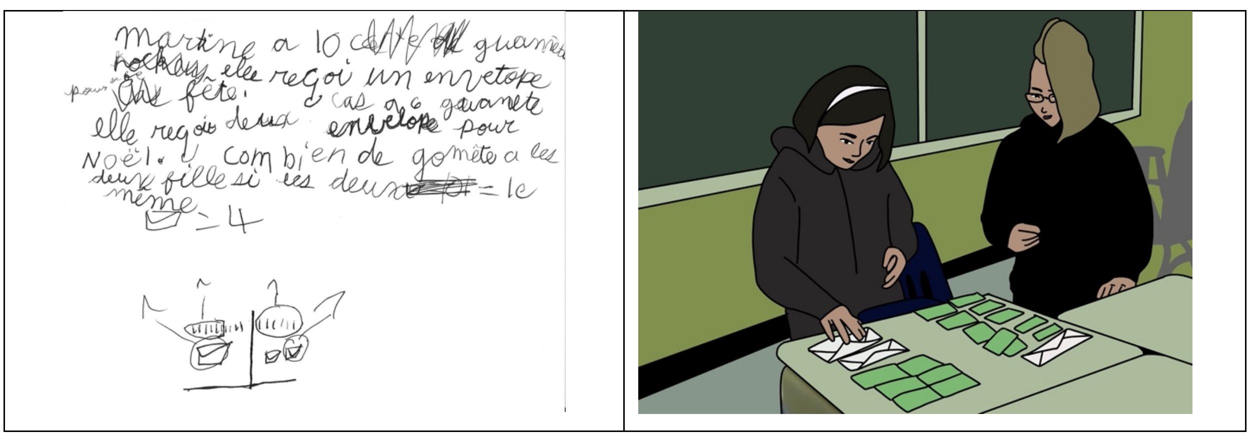

Fig. 8. Inventing a story, and translating and solving it algebraically

The statement of the story bears witness to the students' process of objectification; that is, the process of becoming more and more conscious of the involved concepts, their cultural meanings, and ways of use. Let us consider three structuring elements of the story-problem, namely, the envelopes, the question, and the concept of equal.

The envelopes:

The students did not mention the condition that makes the rule $\mathcal{R}(r[1e])$ applicable, namely that the envelopes must have the same number of stickers.

The question:

The question of the story-problem was not asked in terms of finding the number of stickers in an envelope. The question asked was about the quantity of cards that the girls have.

The concept of equal:

The most difficult part in inventing, translating, and solving the equation was to verbally express the equality of the equating parts. Here is a passage from the students' discussion:

1. Elsa: Si les deux égalent comme ça, you get it, tu comprends? You get it? . . . Combien de gommettes ont les deux filles si les deux ont comme l'affaire égal l'affaire . . . get it? [If both equal like this, you get it, understand? You get it? . . . How many stickers do the two girls have if both have like the matter (one equating part) equal the matter (the other equating part) . . . get it?]

At this point of concept formation, it is not easy for the students to put the equality into words. The previous sections have shown that the relational meaning of equality appears clearly in action. It is *seen* in the concrete objects that the students put on the desk. Yet, here, in inventing a story, its articulation in language is far from easy. In Line 1 equality appears first as a verb ("égalent," present tense of the third person plural), then as an adjective ("égal"). The students were not satisfied with the verbal formulation of equality in the story (acknowledged, among others, by the recurrent "you get it?" uttered in English). In light of this difficulty, the students opted to write the equal sign (=) and "the same" (see Figure 8.1), as if putting both together could help fill up the hole left behind by the awkward linguistic sentence.

The genetic analysis allows us to see the tremendous complexity behind talking about the equal sign. If its relational sense appears clearly in action and perception, it does not in language. Action and perception precede language until the point at which they will merge and form a new psychic unity in concept formation.

4.6 The closing general discussion

To end the teaching-learning activity, the teacher discussed with the class the challenging equation $3x + 1 = 5 + x$. She asked Mimi to solve it. Mimi got up and timidly walked to the board with the activity sheet. The teacher suggested that the activity sheet was unnecessary and, putting an arm on Mimi's shoulder (Figure 9.1), encouraged her to explain the solution. Having noticed that the question about the number of cards in the envelopes was not addressed in several groups, the teacher started calling attention to this important condition:

1. Teacher: (*talking to the class*) First, I ask a question. Does each envelope have the same number of cards?
2. Students: Yes!
3. Teacher: Now it's a matter of finding out how many cards there are in each envelope.
4. Mimi: I remove, I remove one (*she removes a card from the left side*) [and] I remove another one here (*removes a card from the right side*).
5. Teacher: Is your equation still equal?
6. Mimi: Yes.
7. Teacher : YES! (*talking to the class*). Because, what Mimi did, she removed one card on one side (*pointing to the left*) and she did the same thing on the other side (*pointing to the

right) so the equation is still equal (*she moves her arms to convey the idea of a balance in equilibrium; see Figure 9.2*).

8. Mimi: I removed an envelope (*she removes an envelope from the left*) … I have to remove another (*removes envelope from right side*) so that it's equal.
9. Teacher: Yes. So you're telling me that for it to be equal, you have to do it on the other side, so the equation is still equal.
10. Mimi: Afterwards, instead of . . . removing one like that (*points to an envelope*) . . . because we don't have another envelope there (*points to right side*) . . . we have to keep the two envelopes . . . so I divided them in two.
11. Teacher: Why did you divide them in 2?
12. Mimi: Because . . . because I didn't have another envelope to remove (*points to the right side of the equation*) . . . I divided them together, like dividing them in 2 (*she makes a line with her hand between the cards to divide them; see Figure 9.3*).
13. Teacher: To create 2 equal groups!
14. Mimi: 2 equal groups, and then after that, I put 2 (*she moves a card and points to 2 cards*) this is 2 here in this one (*points to an envelope*) and this is 2 here (*points to the other 2 cards*) in the other envelope.

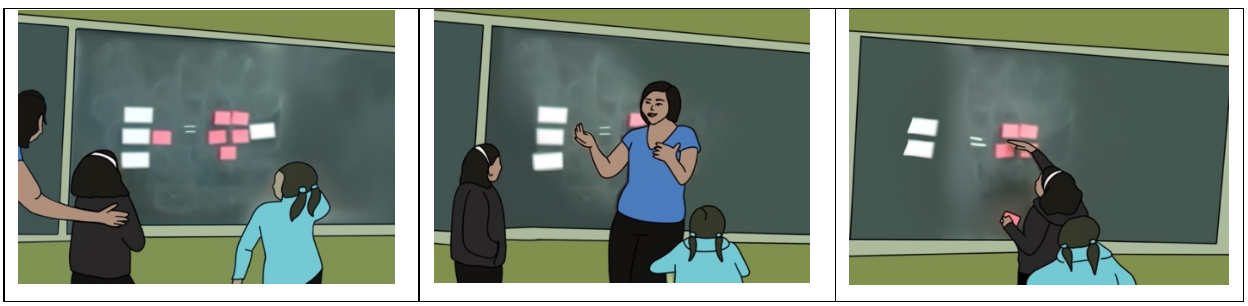

Fig. 9. The closing general discussion

Now, Mimi uses the term "divide" to name the operation. Drawing on Mimi's idea, in Line 13, the teacher offered an interpretation in terms of making equal groups, which connects the new operation to things that the class had previously discussed around the concept of division of numbers.

5. Synthesis and concluding remarks

A great deal of research has shown that students tend to understand equations and the equal sign in procedural terms. The procedural understanding of these concepts overshadows the relational understanding of the equated terms, which is crucial in solving equations through algebraic procedures. The persistence of procedural understandings has led to a call for pedagogical actions that can offer the students opportunities to conceive of equations and the equal sign in relational terms (Carpenter et al., 2003; Matthews & Rittle-Johnson, 2009; Stephens et al., 2013). In responding to this call, in this paper the focus was on investigating the processes through which the students start shifting to the relational understanding of equations and forming the concepts involved in the algebraic simplification of equations. Following Vygotsky's idea about concept formation, the investigation was carried out through a genetic analysis; that is, an analysis that intends to reveal the concepts *in motion*, in the "process of [their] genesis" (Vygotsky, 1997, p. 71). In the context of school learning, that which puts concepts in motion is the *classroom activity*. We drew on the theory of objectification (Radford, 2021), in which classroom activity is conceived of as the joint teaching-learning activity of teachers and students. Joint teaching-learning activity figured as our unit of analysis. In other words, it figured as the *explanans*; that is, that which accounts for what is to be explained (the *explanandum*), which, in our case, is the production and formation of algebraic ideas in simplifying equations.

Our teaching-learning activity was oriented by two interrelated pedagogical principles: the creation of a social space of interaction and communication, and the creation of a task based on *story-problems* of increasing conceptual complexity (Radford, 2021).[6] Instead of using problems involving abstract open arithmetic sentences or alphanumeric equations, we devised two visual semiotic systems (the CSS and the ISS) to model the story-problems. These story-problems worked as narratives that organize experience; they allowed the teacher and the students to infuse equations and their terms with contextual meanings. As Bruner reminds us, a story "is vicarious experience" (1990, p. 54). A story has "a narratorial voice" that includes "either 'reports of real experience' or offerings of culturally shaped imagination" (p. 54).

The focus in this paper was on the introductory Grade 3 teaching-learning activity for equations. The activity was geared towards the following:

(1) allowing the students to understand how the story-problems are expressed/translated in the CSS and/or the ISS;

(2) raising an awareness about differences in problem-solving procedures;

(3) emphasizing the relational meaning of the equated parts in an equation; and

(4) gaining a deep understanding of the operation-based rules that play a central role in the simplification of equations.

In the opening general discussion, the students tended to resort to the Computational and the Comparison Procedures. However, through their joint work with the teacher, the students encountered new ways to think of equations in relational terms and to make sense of the operation-based rules that lead to the simplification of equations.

The multimodal semiotic analysis shows that the first rule (i.e., removing one card on both sides of the equation, the rule $\mathcal{R}(r[1c])$) appeared in an embodied way in the first general discussion. The rule appeared through movement, action, tactility, and perception. In its embodied form, the conscious content of the rule rests first of all on the immediately perceived objects (cards and envelopes) and their relations. To ensure a transition to a more developed form of conscious content, with the participation of the teacher, the embodied appearance of the rule was supplemented with a theoretical dimension through language. Indeed, when Cyr starts solving the equation (Line 4, first general discussion), the teacher *names* the operation: the operation is about *removing*. We noted that *naming* (or nominalization) is a central aspect of becoming conscious of something and of concept formation. From there on, the operation was also named by the students. The $\mathcal{R}(r[1c])$ rule was later *generalized* to removing several cards at once—i.e., the $\mathcal{R}(r[nc])$ rule. It offered a background to imagine the fundamental rule $\mathcal{R}(r[1e])$, which involves operating with *unknown quantities* in equating relations. It is in the ability of operating with

[6] The six problems discussed in this paper constituted the task and were chosen following the theoretical guidelines of our approach to task design. One distinctive idea of task design in the theory of objectification is that the problems of the task follow "an organization . . . according to an increasing conceptual complexity" (Radford, 2021, p. 134). Simpler problems come first (e.g., $x + 2 = 8$), complex problems come later (e.g., $3x + 1 = 5 + x$), while what is learned in one problem is put in the service of solving the next problems.

unknown quantities that Filloy and Rojano (1989) saw the transition from arithmetic to algebraic thinking. In the course of their journey, the students also encountered another key rule in simplifying equations, namely the $\mathcal{R}(s[c,e])$ rule, first termed *separating*, then associated with the *division* of things into groups to solve the equation $2x = 4$.

What is the role of the ISS? Is it necessary? The ISS offered context to further the students' "conscious awareness of concepts and thought operations" (Vygotsky, 1987, p. 185). Indeed, the ISS is not a mere replica of the CSS. In the ISS, the spatial-temporal embodied dimension of thinking needs to be expressed through *written signs*. The arrow that was used to represent the sensuous algebraic operation of removing cards and envelopes endows the operation with a *permanence*, a durable presence it did not have in its embodied form, where the embodied operation disappeared as soon as it was performed. Through the written sign, the operation acquired a tangibility beyond the one given to it by the spoken word. It became an object of perception and attention on par with cards and envelopes.

The series of story-problems of increasing conceptual complexity culminated in asking the student to invent a story to be translated and solved in the devised semiotic systems. Through asking the students to invent and write the story, we (the teacher and the research team) sought to strengthen the students' awareness and understanding of the conceptual structure of relational equations, and the operation-based rules to simplify the equations. The story-problems that the students produced were close to the ones they encountered in previous problems; these problems worked as models. To some extent, we could say that the students' stories were imitations of those models. But it is important to seize the cognitive importance of imitation. For one thing, the students' stories are not an automatic copy of the models. As the American psychologist James Mark Baldwin wrote, through imitation, the child

> gets the 'feel' of things that others do . . . he (sic) tries on the varied ways of doing things, and so learns his own capacities and limitations . . . he actually acquires the stored up riches of the social movements of history . . . [and] learns to use the tools of culture, speech, writing, manual skill, so that through the independent use of these tools he may become a more competent and fruitful individual. (Baldwin, 1911, p. 21)

Following Baldwin, Vygotsky (1987) added that "the child can imitate only what lies within the zone of his own intellectual potential" (p. 209). This is why children can imitate certain things but not others. "If I am not able to play chess, I will not be able to play a match even if a chess master shows me how" (Vygotsky, 1987, p. 209).

These capacities and limitations as well as the intellectual potential transpired in the story and solutions that the students produced. In inventing a story, the students assumed the role of the author; they had not only to solve the story-problem through the operation-based rules with which they had become acquainted, but also narratively to articulate the equating parts and the equality, thereby pushing the students' concept formation to new levels. There are things that will require further development, such as a more sophisticated verbal articulation of the equal relation, and the explicit consideration of the condition that the envelopes must have a same number of cards (which was the object of discussion the next day). We see nonetheless that the recognition of the relational meaning of the equal sign and the application of the rules of simplification of equations have become progressively understood and applied with ease.

What are the contributions of this paper? By unveiling a promising route for introducing students to equations in early algebra, this paper makes a contribution to instruction and curriculum development—curriculum understood as a cultural practice (Yatta, 2002). Indeed, the semiotic systems presented here (the CSS and the ISS) provide teachers and curriculum designers with valuable instructional ideas. As this paper has shown, these semiotic systems are bearers of great potentiality in concept formation. Yet, a semiotic system, much as a tool, does not make sense in itself. A semiotic system is really helpful in a learning context when it is deployed and used in an activity susceptible of maximizing its cognitive potential. It is at this point that the theoretical concept of teaching-learning activity discussed in this paper can be considered. The paper also makes a contribution to mathematics education research. It distinguishes itself from other studies in its focus: the research problem is not about *what* students *think* of equality and related concepts in solving equations (a typical psychological problem, often tackled through questionnaires and interviews), but *how* students *learn in situ*, in the classroom, (the educational problem *par excellence*). In this sense the fine-grained genetic analyses of concept formation that were presented here reveal some of the challenges that teachers and students face in teaching and learning the concepts that underlie the processes of solving equations algebraically. The historical-epistemological analysis of Arabic algebra presented in Section 2.1 shows the thorough and careful terminology Arabic mathematicians developed to talk and think about equality. This analysis is certainly a reminder that equality is a complex mathematical concept—and one we cannot get rid of.[7] As the mathematician Barry Mazur (2008, p. 222) noted, "One can't do mathematics for more than ten minutes without grappling, in some way or other, with the slippery notion of *equality*." Equality is an omnipresent concept with various facets, bearing different meanings and requiring different understandings.

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